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EMBEDDABILITY OF PTYKES

by

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Embeddings (or natural transformations) are the morphisms of ptykes. The embeddability relation "there is a morphism from A to B" plays a great role in the theory of ptykes, and this paper investigates some basic properties of this relation.

(i) When there is an embedding from A to B, then there is a smallest one; but this does not mean that this morphism has a simple construction. The theory of weak morphisms of section II gives a simple inductive way of constructing this morphism (if it exists) or to reckognize that it does not exist otherwise. One utilisation of this theory can be found in section III, where weak morphisms are used to reduce embeddability problems to the denumerable case.

(ii) When F is a sufficiently definable function from ptykes of type σ to ptykes of type τ, then it is possible to bound F by a recursive ptyx Φ. Here the bounding is pointwise, w.r.t. the embeddability relation. In section III, we prove two of these results, one for Σ^1_1 functions and denumerable arguments, the other for set-recursive functions.

(iii) The main technical tool in the functorial bounding theorems of section III are the amalgamation results of section I. These results enable us to complete certain (non directed) inductive systems of ptykes of type σ. The completion (amalgamation) is not functorial in all the data, but enough functoriality is left for applications.

(iv) The problem of embedding a family of ptykes is the following : we have a set X of ptykes of type σ ; then find a sufficiently regular family Y such that any element of X embeds in some element of Y. Kechris & Woodin have embedded in [2] Σ^1_1 sets of ptykes into a single recursive one. In section IV we extend this result to the case where X is Σ^1_k, and Y is the set $\{D(a); a \text{ of type } k-2\}$ generated by a recursive D of type $k-2 \to \sigma$. For this result, we use T_A-technology, which relates embeddability of A with embeddability of the functorial tree T_A of descending sequences of A.

(v) Finally we solve in an appendix a problem of Kechris : we show that for some recursive D, the set of ordinals $D'(\omega_1)$, where D' is embeddable in D, is not denumerable.

BACKGROUND

some familiarity with [1] (esp. chapter XII) can be helpful. Recall that if we consider the finite types built from O (ordinals) and () (unit category) by means of the operations $\rightarrow$ and X, then any type is isomorphic to a product of connected types of the form $\sigma \rightarrow O$. Most of the results we consider are trivial for products if we assume them for the components. We recall the notations $|\sigma|$ = objects of type σ (ptykes of type σ), $I(a,b)$ = set of morphisms from a to b.

There are many possible notions of recursive ptyx; here we shall only consider the most liberal one, i.e. a ptyx is recursive when it is a recursive direct system of finite dim ensional ptykes. (*)

The functor Trace is the main tool in the study of ptykes ; the trace is inductively defined as follows :
if A is of type $\sigma \rightarrow 0$, then we have the normal form theorem for elements of $A(a)$ ($a \in |\sigma|$) : $z \in A(a)$ can be written as $z = (z_0;a_0;t;a)_A$; this means that
(i) $t \in I(a_0,a), z_0 \in A(a_0)$ and $z = A(t)(z_0)$
(ii) among all solutions of (i), t is chosen with $rg(Tr(t))$ minimal.

We have existence and unicity for normal forms. Tr(A) is the set of all possible components (z_0,a_0) occuring in A-normal forms. When $T \in I(A,B)$, Tr(T) maps Tr(A) into Tr(B) by $Tr(T)(z_0,a_0) = (T(a_0)(z_0),a_0)$. One of the characteristic properties of the trace is that if we know B, and the subset $rg(Tr(T))$, then T and its source A are completely determined; in particular, B and Tr(T) determine T.

The reader can also consult [5] where ptykes are viewed as logical structures, although the formalism of this paper has not been followed here.

(*) a ptyx is finite dimensional when its dimension i.e. the cardinality of its trace if finite. Any ptyx is a direct limit of finite dimensional ptykes.

I AMALGAMATIONS

I.1. definition

Let I be a non void partial order ; an inductive system (indexed by I) of ptykes of type σ is a family (A_i, T_{ij}) with :

- A_i ptyx of type σ for all $i \in I$
- $T_{ij} \in I(A_i, A_j)$ for all $i,j \in I$ with $i \leq j$, $T_{js}T_{ij} = T_{is}$ when $i \leq j \leq s$

An amalgamation for (A_i, T_{ij}) is a family (A, T_i) such that

- A is a ptyx of type σ
- $T_i \in I(A_i, A)$ for all $i \in I$
- for all $i,j \in I$ with $i \leq j$, $T_j T_{ij} = T_i$

So an inductive system is the same thing as a direct system, except that we do not require I to be directed. It is possible to define a concept of inductive limit, but in the categories we are dealing with, inductive limits essentially exist when I is directed (direct limits). This is the reason for considering amalgamations, which are non canonical ways of gluing together families of ptykes. There are two typical questions that are connected to amalgamations :

(i) Find reasonable hypotheses ensuring the existence of an amalgamation for (A_i, T_{ij})

(ii) Find reasonable situations where the amalgamations constructed when solving (i) are functorial.

Typical examples of amalgamations are given by :

I.2. examples

i) I is the disjoint sum of a singleton {u} and an ordinal x, and is ordered by $u \leq u$, $i \leq i$ $(i \in x)$, $u \leq i$ $(i \in x)$. In [1] (chapter XII) an amalgamation is constructed for inductive systems indexed by such I's. Of course the construction depends on the ordinal structure of x (we must wellorder I).

ii) I is the set of finite sequences of integers, ordered by $s \leq s*s'$ (the opposite of the usual tree relation), and the inductive system is a recursive direct system of finite dimensional ptykes. In [2] a recursive amalgamation for such a system is constructed. Observe that the only hypothesis is that if one restricts oneself to any linear subset (branch) of I, then the direct limit $\lim\limits_{\vec{B}}(A_i, T_{ij})$ exists, and this hypothesis is clearly necessary. In particular the (non denumerable) set X consisting of all such direct limits has the property that any element $a \in X$ can be embedded into our amalgamation A, and this is the way one proves that any Σ^1_1 set of denumerable ptykes is bounded (w.r.t. embeddability) by a recursive ptyx of the same type.

These two examples suggest us to restrict oneself to optrees : I is an optree when the opposite of I is a tree, namely :

- I has a smallest element bot(I)

- for any $i \in I$, the set $\{j; j <_I i\}$ is linearily ordered by I, and finite. When $i \in I$, let $dpth(i) = card\{j; j <_I i\}$, and when $i \neq bot(I)$, let $i^* = sup\{j; j <_I i\}$.

The amalgamations we shall construct are non canonical ; they do depend on some auxiliary wellordering R of $|I|$, defined as follows : for each $i \in I$, choose a wellorder R^i on the set $|R^i| = \{j \in I; j^* = i\}$; we define $i <_R j$ to mean that either $dpth(i) < dpth(j)$ or $dpth(i) = dpth(j)$ and $i' <_R k\ j'$, with $k = inf(i,j)$ and i', j' are defined by $i'^* = j'^* = k$, $i' \leq_I i$, $j' <_I j$.

I.3. theorem

If (A_i, T_{ij}) is an inductive system of ptykes of type σ, indexed by an optree I, and if $\lim\limits_{\vec{L}}(A_i, T_{ij})$ exists for any linear subset of I, then one can define (depending on the choice of some wellorder R, defined as above) an amalgamation $(A, T_i) = \underset{\vec{R}}{\mathrm{amalg}}(A_i, T_{ij})$ for (A_i, T_{ij}).

PROOF : this is by induction on the type σ; up to isomorphism, σ is a finite product of types $\tau_i \to 0$, and the theorem for a product follows from the theorem for the components : we shall therefore prove the result for $\sigma = \tau \to 0$, assuming it for the type τ.

If $a \in |\tau|$, define a set $|A(a)|$ by :

$|A(a)| = \{(i,z); i \in |I| \ \& \ z \in |A_i(a)| \ \& \ (i = \mathrm{bot}(I) \text{ or } z \notin \mathrm{rg}(T_{i*i}(a)))$

When $i \in I$, define a function $T_i(a)$ from $|A_i(a)|$ to $|A(a)|$ by :

$T_i(a)(z) = (j,z')$ where $j \leq_I i$ & $z = T_{ji}(a)(z')$. Clearly $T_j(a)T_{ij}(a) = T_i(a)$.

If $t \in I(a,b)$, one can define a function $A(t)$ from $|A(a)|$ to $|A(b)|$, by $A(t)(i,z) = (i, A_i(t)(z))$. Now it is easy to prove a normalform theorem for the points of $|A(a)|$, and from this it follows that our problem of amalgamation reduces to the definition of wellorders $A(a)$ on the sets $|A(a)|$ such that :

(i) $T_i(a) \in I(A_i(a), A(a))$

(ii) $A(t) \in I(A(a), A(b))$ when $t \in I(a,b)$

We can define a binary (non transitive) relation $B(a)$ on $|A(a)|$: $(i,z) <_{B(a)} (j,z')$ iff either $i \leq_I j$ and $T_{ij}(a)(z) <_{A_j(a)} z'$ or $j \leq_I i$ and $z <_{A_i(a)} T_{ji}(a)(z')$. Then $x <_{A_i(a)} y \rightarrow T_i(a)(x) <_{B(a)} T_i(a)(y)$, so we can replace (i) by :

(i)' $A(a)$ extends $B(a)$

The orders $A(a)$ are defined simutaneously as follows : if we want to compare (i,z) with (j,z'), then we can assume that $i \neq j$ because when $i = j$, $B(a)$ already yields a comparizon. Then by symmetry, we can assume that $i <_R j$:
either one can find $b \in |\tau|$, $t \in I(a,b)$, $k <_I j$, $(k,z'') \in |A(b)|$ such that $(j, A(t)(z')) <_{B(b)} (k,z'') \leq_{A(b)} (i, A(t)(z'))$: in this case $(j,z') <_{A(a)} (i,z)$.
otherwise, we say that $(i,z) <_{A(a)} (j,z')$.

This is a definition by transfinite induction (modulo R) : in order to compare (i,z) and (j,z') in $A(a)$, we need to know how to compare (k,z'') and $(i, A(t)(z))$ in $A(b)$; remarking that $\sup(i,j) >_R \sup(k,i)$ it is easy to transform this definition into a definition by transfinite induction.

<u>I.4. lemma</u>

The relations $A(a)$ are strict order relations, enjoying conditions (i) and (ii).

PROOF : condition (i) holds : it suffices to check (i)', i.e. that $(i,z) <_{B(a)} (j,z') \to (i,z) <_{A(a)} (j,z')$, and this property is easily established by induction on $\sup(i,j)$:

- if $j <_I i$, then (with $b = a$, $t = E_a$, $z'' = z'$, $k = i$) $(i,z) <_{A(a)} (j,z')$
- if $i <_I j$, but $(j,z') <_{A(a)} (i,z)$, choose $b,t,z'',k <_I j$ such that $(j,A(t)(z')) <_{B(b)} (k,z'') \leq_{A(b)} (i,A(t)(z))$; but $i,k <_I j$, so they are comparable for $\leq_I$, and so (k,z'') and $(i,A(t)(z))$ are comparable for $\leq_{B(b)}$, and by the induction hypothesis, the only possibility is $(k,z'') \leq_{B(b)} (i,A(t)(z))$. Now observe that $\leq_{B(b)}$ is transitive when the indices (here : j,k,i) are pairwise comparable, so $(j,A(t)(z')) <_{B(b)} (i,A(t)(z))$... contradiction.

condition (ii) holds : if $(i,z) <_{A(a)} (j,z')$, then $(i,A(t)(z)) <_{A(b)} (j,A(t)(z'))$: by induction on $\sup(i,j)$:

- when $i = j$ this is because the property holds with B instead of A
- when $i \lessdot_R j$, the result is immediate
- when $j \lessdot_R i$, if c,u,z'',k are such that $k <_I i$ and $(i,A(u)(z)) <_{B(c)} (k,z'') \leq_{A(c)} (j,A(u)(z'))$, it is possible to apply the theorem I.3. (induction hypothesis) to τ, as in example I.2., with $x = 2$: we get d, $t' \in I(b,d)$, $u' \in I(c,d)$ such that $t't = u'u$. Then $(i,A(t't)(z)) <_{B(d)} (k,A(u')(z)) \leq_{A(d)} (j,A(t't)(z'))$ (using the induction hypothesis on (k,j) and function u') so $(i,A(t)(z)) <_{A(b)} (j,A(t)(z'))$.

Now $A(a)$ is obviously linear and irreflexive ; we check transitivity : if $(i,z) <_{A(a)} (j,z') <_{A(a)} (k,z'')$, we prove that $(i,z) <_{A(a)} (k,z'')$ by induction (modulo R) on $\sup(i,j,k)$:

<u>1</u> if $i = j = k$ the problem is a problem of transitivity for $B(a)$; but we have already observed that $B(a)$ is transitive when i,j,k are pairwise comparable for $\leq_I$.

<u>2</u> if $k <_R i = j$, then one can find b,t,z''',k' such that $(j,A(t)(z')) <_{B(b)} (k',z''') \leq_{A(b)} (k,A(t)(z''))$; but then $(i,A(t)(z)) <_{B(b)} (k',z''')$, so $(i,z) <_{A(a)} (k,z'')$.

<u>3</u> if $j,k \lessdot_R i$, then for appropriate $b,t,z''',k' <_R i$, one has

$(i,A(t)(z)) <_{B(b)} (k',z''') <_{A(b)} (j,A(t)(z')) <_{A(b)} (k,A(t)(z''))$

(the last inequality comes from property (ii)); the induction hypothesis yields $(k',z''') <_{A(b)} (k,A(t)(z''))$, so $(i,z) <_{A(a)} (k,z'')$.

4 there are four remaining cases, namely $i <_R j = k$, $j <_R i = k$, $i,j <_R k$, $i,k <_R j$: each of these cases can easily be reduced to case 2 or case 3, for instance if $i <_R j = k$, we argue by contradiction : if $(i,z) \not<_{A(a)} (k,z'')$ then $(k,z'') \leq_{A(a)} (i,z)$, and if we apply case 2 to $(j,z'),(k,z''),(i,z)$, we obtain $(j,z') <_{A(a)} (i,z)$, a contradiction. ∎

I.5. lemma

Assume that $(i,z) <_{A(a)} (j,z')$ and $j <_R i$; then one can find b, $t \in I(a,b)$, $(k,z'') \in |A(b)|$ such that

$$(i,A(t)(z)) <_{B(b)} (k,z'') \leq_{A(b)} (j,A(t)(z')) \qquad (1)$$

and $k <_I i$, $k \leq_R j$.

PROOF : compared to the definition of $<_{A(a)}$, the only difference is the additional requirement $k \leq_R j$; we prove the result by induction (in R) on $\sup(i,j)$: if $(i,z) <_{A(a)} (j,z')$, then one can find $k <_I i$, b, t, z'' such that (1) holds ; if $k \leq_R j$, we are done. Otherwise, remark that $\sup(k,j) <_R \sup(i,j)$, and the induction hypothesis yields $c,u \in I(b,c)$, $k' <_I k$, such that $(k,A(u)(z'')) <_{B(c)} (k',z''') \leq_{A(c)} (j,A(ut)(z'))$ and $k' \leq_R j$. But then

$(i,A(ut)(z)) <_{B(c)} (k,A(u)(z'')) <_{B(c)} (k',z''') \leq_{A(c)} (j,A(ut)(z'))$

and using a transitivity argument for B(c), we see that c,ut,k',z''' enjoy the property. ∎

I.6. lemma

The relations A(a) are wellorders.

PROOF : it is enough to check that there is no s.d.s. (strictly decreasing sequence) for A(a). We start with such a s.d.s. (i_n,z_n) ; we shall successively show that we can assume stronger and stronger hypotheses on (i_n,z_n) (this means that a subsequence of (i_n,z_n) enjoys these stronger properties), up to a contradiction :

(i) the sequence (i_n) is strictly incresing for $<_R$: by a Ramsey argument one can form a subsequence (j_n, z'_n) such that one of the following holds :

1 $j_{n+1} <_R j_n$ for all n

2 $j_{n+1} = j_n$ for all n

3 $j_n <_R j_{n+1}$ for all n

1 is impossible because R is a wellorder

2 is impossible because this would yield a s.d.s. of the form (i, z'_n) for $B(a)$, and this sequence would yield a s.d.s. (z'_n) in $A_i(a)$.

So the only possibility is 3. We assume that (i_n, z_n) enjoys 3.

(ii) one of the following holds

1 the sequence (i_n) is strictly increasing for $<_I$

2 there is an index i such that for all n,m $\inf(i_n, i_m) = i$ (if $n \neq m$).

This new condition is realized as follows : we consider the points $i \in |I|$ such that $\{n; i <_I i_n\}$ is infinite. If the set X of these points contains a maximal element for $<_I$, then a subsequence of (i_n, z_n) enjoys 2. Otherwise a subsequence enjoys 1.

(iii) the possibility (ii)1 is absurd : since the i_n's are pairwise comparable for $<_I$, our s.d.s. is a s.d.s. for $B(a)$; moreover, if $L = \{i_n; n \in \mathbb{N}\}$, L is a linear subset of $|I|$, and (i_n, z_n) can be viewed as a s.d.s. in the direct limit $\varinjlim_L (A_i(a), T_{ij}(a))$, contradicting the assumptions.

(iv) so we are in situation (ii)2 ; define j_n by $j^*_n = i$ and $j^*_n \leq_I i_n$. Then we can assume that (j_n) is strictly increasing (similar to restriction (i)).

(v) Let n be an integer; define 4-tuples (b_p, t_p, k_p, x_p) such that $b_p \in |\tau|$, $t_p \in I(a, b_p)$, $x_p \in |A(b_p)|$ and when $p \neq 0$, $k_p <_I i_{n+p}$ and $(i_{n+p}, A(t_p)(z_p)) <_{B(b_p)} (k_p, x_p) \leq_{A(b_p)} (i_n, A(t_p)(z_n))$.

For $p = 0$, let $b_0 = a$, $t_0 = E_a$, $k_0 = i_n$, $x_0 = z_n$. If (b_p, t_p, k_p, x_p) has been defined and $k_p \leq_I i$, then the next 4-tuple is not defined ; otherwise, we apply lemma I.5. to the inequality $(i_{n+p+1}, A(t_p)(z_{p+1})) <_{A(b_p)} (k_p, x_p)$: we get $b_{p+1}, u_p \in I(b_p, b_{p+1})$, $k_{p+1} <_I i_{n+p+1}$ and z_{p+1} such that $k_{p+1} \leq_R k_p$ and $(i_{n+p+1}, A(u_p t_p)(z_{p+1})) <_{B(b_{p+1})} (k_{p+1}, z_{p+1}) \leq_{A(b_{p+1})} (k_p, A(u_p)(z_p))$.

Then $t_{p+1} = u_p t_p$ defines the next 4-tuple.

The inequality $k_{p+1} \leq_R k_p$ implies $dpth(k_{p+1}) \leq dpth(k_p)$; but these depths cannot be equal, because $j_p \leq_I k_p$, $j_{p+1} \leq_I k_{p+1}$ and $dpth(k_p) = dpth(k_{p+1})$ implies (by property (iv)) $k_p <_R k_{p+1}$. So there is an integer p such that $dpth(k_p) \leq dpth(i)$; but $i, k_p <_I i_p$, so $k_p \leq_I i$.

Summing up we have obtained, starting with n, an integer $m > n$, an object b of $|\tau|$, $t \in I(a,b)$, $k \leq i$ and x such that

$$(i_m, A(t)(z_m)) <_{B(b)} (k,x) \leq_{A(b)} (i_n, A(t)(z_n))$$

(vi) using (v), we can assume that we have together with (i_n, z_n), a family (b_n, t_n, k_n, x_n) such that

$$(i_{n+1}, A(t_n)(z_n)) <_{B(b_n)} (k_n, x_n) \leq_{A(b_n)} (i_n, A(t_n)(z_n))$$

for all n : simply iterate (v) and extract a subsequence.

(vii) apply amalgamation at type τ (example I.2.i)), to get c and $u_n \in I(b_n, c)$ such that $u_n t_n = u_m t_m$ for all n,m. Then $(k_n, A(u_n)(x_n))$ is a s.d.s. for B(c) (the k_n's being $\leq_I i$ are pairwise comparable) ; this s.d.s. is isomorphic to a s.d.s. in $A_i(c)$, yielding the final contradiction. ∎

This lemma ends the proof of the theorem. ∎

I.7. remarks

(i) if $(A, T_i) = \underset{R}{\mathrm{amalg}}(A_i, T_{ij})$, if L is a linear subset (more generally : directed) of $|I|$, then consider $(B, U_i) = \underset{L}{\lim}(A_i, T_{ij})$: by the universal property of direct limits, there is a unique $V \in I(B,A)$ such that $VU_i = T_i$ for all $i \in L$. In particular, this shows that all the dirtect limits along linear subsets of I are embeddable in A.

(ii) amalgamation is functorial in the following sense : define a category IND^σ (for "inductive systems") by :

objects : 3-tuples $(I, R, (A_i, T_{ij}))$ where I is an optree, R is a linearization of I as considered in I.3., and (A_i, T_{ij}) is an inductive system of ptykes of type σ, indexed by I, enjoying the conditions of I.3. .

morphisms from $(I, R, (A_i, T_{ij}))$ to $(J, S, (B_l, U_{lm}))$: all injective functions h from $|I|$ to $|J|$ such that :

- $h(bot(I)) = bot(J)$
- for $i \neq bot(I)$, $h(i^*) = h(i)^*$
- $i <_R j \to h(i) <_S h(j)$
- $B_{h(i)} = A_i$, $U_{h(i)h(j)} = T_{ij}$ $\qquad (i,j \in |I|,\ i \leq_I j)$

Then the amalgamation process of I.3. can be viewed as a functor Φ from IND^σ to σ: define $\Phi(I,R,(A_i,T_{ij})) = \underset{R}{\mathrm{amalg}}(A_i,T_{ij})$, as in I.3. ; when h is a moprhism from $(I,R,(A_i,T_{ij}))$ to $(J,S,(B_l,U_{lm}))$, then, up to isomorphism $(I,R,(A_i,T_{ij}))$ can be viewed as a substructure of $(J,S,(B_l,U_{lm}))$; the same is true for their respective amalgamations (A,T_i) and (B,U_l) : going back to the definition of $A(a)$, $B(a)$ in I.3., we see that when $l,m \in J$, the way we compare (l,z) and (m,z') only depends on the restriction of $(B_.,U_{..})$ to $\{k; k \leq_S l$ or $k \leq_S m\}$ as well as the restriction of the order S to this set. In the case $l,m \in I$, we see that the data that enable us to compare (l,z) with (m,z') are the same, whether we work in $A(a)$ or in $B(a)$. From this it follows that $A(a)$ can be viewed as a substructure of $B(a)$ (and more generally A of B) : there is therefore a unique morphism $V \in I(A,B)$, such that $VT_i = U_{h(i)}$ for all $i \in I$, and this morphism $\underset{h}{\mathrm{amalg}}(J,S,(B_l,U_{lm}))$ is by definition $\Phi(h)$. One checks without problems that Φ preserves direct limits and pull-backs.

(iii) a more general notion of morphism would be that of an inductive system of morphisms : typically in (ii) instead of $U_{h(i)h(j)} = T_{ij}$, require the data of $V_i \in I(A_i,B_{h(i)})$ with $V_jT_{ij} = U_{h(i)h(j)}V_i$: amalgamation is not functorial w.r.t. this extended notion.

(iv) amalgamation is not effective in general. However, when the A_i's are finite dimensional, then the amalgamation will be effective in the data (i.e. $|I|$, $bot(I)$, the function $i \to i^*$, R, the system (A_i,T_{ij})). By the functoriality remark of (ii), it suffices to show that the amalgamation is effective in the data when I is a finite set.

Let us first consider the case $I = \{0,1,2\}$ with $0 <_I 1,2$, linearized into R by $1 <_R 2$. Starting with an inductive system of finite dimensional ptykes (A_i, T_{ij}), we construct (A, T_i) as in I.3. ; A is a finite dimensional ptyx $(\dim(A) = \dim(A_1) + \dim(A_2) - \dim(A_0))$, and we have to indicate how it may be effectively computed from (A_i, T_{ij}). So take $a \in |\tau|$, a finite dimensional, and let us see how to compare points (i,z), (j,z') in $|A(a)|$: when i and j are comparable for $\leq_I$, then (i,z) and (j,z') both belong to $rg(T_{1}(a))$ or to $rg(T_{2}(a))$, and so we already know how to compare them. So we are reduced to compare $(1,z)$ with $(2,z')$, and by definition $(1,z) <_{A(a)} (2,z')$ iff for some b, for some $t \in I(a,b)$, for some $z'' \in |A_0(b)|$: $A_1(t)(z) <_{A_1(b)} T_{01}(b)(z'')$ and $T_{02}(b)(z'') <_{A_2(b)} A_2(t)(z')$

If we can show that we can limit our search for b,t,z" to a finite set, then we shall get a decision procedure : first z" must have a normalform $(z_i;a_i;u;b)_{A_0}$, and since A_0 is finite dimensional, we can successively try all possible pairs $(z_i;a_i) \in Tr(A_0)$; so let us fix an element $(z_p;a_p)$ of $Tr(A_0)$; then our search has been reduced to the search of $b, t \in I(a,b), u \in I(a_p,b)$; it is easy to show that one can require $Tr(b) = rg(Tr(t)) \cup rg(Tr(u))$; but the general results of [1] , chapter XII, establish that the set of such b,t,u is finite, and effective in the data a_0,a.

More generally, we must solve the problem when I is arbitrary : I has a greatest element i for $<_R$; if i is also the grestest element for $<_I$ then $A = A_i$ etc. ; otherwise let $J = I - \{i\}$, S the restriction of R to J, and $(A', T'_i) = \underset{S}{\mathrm{amalg}}(A_i, T_{ij})$. We define $B_0 = A_{i*}$, $B_1 = A'$, $B_2 = A_i$, $U_{01} = T'_{i*}$, $U_{02} = T_{i*i}$. Then the amalgamation of (B_1, U_{1m}) is exactly equal to the amalgamation of (A_i, T_{ij}), and (B_1, U_{1m}) can be effectively amalgamed : so we finally obtain an effective way of constructing A by induction on $card(|I|)$.

II WEAK MORPHISMS

Let A and B be ptykes of the same type σ . We are seeking an inductive procedure that will "decide" the existence of a morphism from A to B. We shall proceed by ordinal stages α , and at each stage, it will be possible to reckognize from what has already been done that we are in one of the three following situations :

<u>1</u> the process is not mature, i.e. go on

<u>2</u> the process cannot be continued further, i.e. there is no morphism

<u>3</u> we have got the morphism

When the type σ is a product $\sigma_1 \times \ldots \times \sigma_p$, then the existence of a morphism T from $A_1 \otimes \ldots \otimes A_p$ to $B_1 \otimes \ldots \otimes B_p$ is equivalent to the existence of morphisms $T_1 \in I(A_1,B_1),\ldots,T_p \in I(A_p,T_p)$, so we can restrict to the case where the type σ is a connected type $\tau \to 0$.

II.1. definition

Let A,B be ptykes of type $\tau \to 0$, and let $a \in |\tau|$. A weak morphism from A to B at stage a is a function $h \in I(A(a),B(a))$ such that :

$$\forall z \in A(a) \; (\mathrm{Den}_{B,a}(h(z)) \subset \mathrm{Den}_{A,a}(z))$$

II.2. example

If $T \in I(A,B)$, then T(a) is a weak morphism from A to B at any stage a.

II.3. remark

By induction on B(a), it is possible to construct a weak morphism from A to B at stage a, or to reckognize that there is no such object : define

$$h(z) = \inf \{ z';\mathrm{Den}_{B,a}(z') \subset \mathrm{Den}_{A,a}(z) \;\&\; \forall t \; (t < z \to h(t) < z')\}$$

If there is a weak morphism at stage a, then this formula defines the smallest one. If for some z, h(z) cannot be defined, then there is no weak morphism.

II.4. proposition

Assume that $t \in I(a,b)$ and that h is a weak morphism from A to B at level b; then it is possible to find a weak morphism k at level a such that the diagram

$$\begin{tikzcd} A(a) \arrow[r, dashed, "k"] \arrow[d, "A(t)"'] & B(a) \arrow[d, "B(t)"] \\ A(b) \arrow[r, dashed, "h"'] & B(b) \end{tikzcd}$$

is commutative. Notation : $k = t^{-1}(h)$.

PROOF : if $z \in A(a)$, we define $z' = k(z)$ by : $B(t)(z') = h(A(t)(z))$. The problem is to show that this can be done.

But $Den_{B,b}(h(A(t)(z))) \subset Den_{A,b}(A(t)(z)) \subset rg(Tr(t))$

so $h(A(t)(z)) \in rg(B(t))$. The function k is uniquely determined and is order preserving. The fact that it is a weak morphism is immediate. ∎

II.5. remark

The concept of weak morphism corresponds to another possible concept of morphism of ptykes. Like natural trasformations, they can be restricted (if we know $T(b)$ then we can build $T(a)$ from any $t \in I(a,b)$), but the restriction depends on the choice of $t \in I(a,b)$. But also natural transformations can be extended (e.g. if A and B are dilators, then from $T(\omega)$, we can recover $T(x)$ for all x), and this is not the case with weak morphisms. The idea of weak morphism is therefore a concept of "limited morphism" (at stage a).

II.6. theorem

There is a natural trasformation from A to B iff for all $a \in |\tau|$ there is a weak morphism from A to B at stage a.

PROOF : the "only if" part of the theorem is just example II.2. . Conversely, suppose that there is a weak morphism at each stage a, and let h_a be the smallest one (defined by remark II.3.). If $t \in I(a,b)$, we use the notation $h_a^{t,b}$ to denote $t^{-1}(h_b)$. $h_a^{t,b}$ is a weak morphism at stage a, and the question is to compare $h_a^{t,b}$ with $h_a^{u,c}$: when $u = t't$ for some $t' \in I(b,c)$ then $h_a^{u,c} = t^{-1}(t'^{-1}(h_b)) \geq t^{-1}(h_b) = h_a^{t,b}$. Futhermore, given any family (t_i, b_i) with $t_i \in I(a,b_i)$, one can find (u_i,c) such that $u_i t_i$ = constant (amalgamation, see fro instance II.2.i)). This establishes that the set of weak morphisms $h_a^{t,b}$, when t and b vary, has a greatest element h_a^*.

Assume that $t \in I(a,b)$; if $h_a^* = h_a^{u,a'}$, $h_b^* = h_b^{v,b'}$, then $t^{-1}(h_b^*) = h_a^{vt,b'}$; if $w,w' \in I(b',c)$ are such that $wu = w'vt$, then $h_a^* = h_a^{wu,c}$ $(= h_a^{w'vt,c})$ by maximality, similarily, $h_b^* = h_b^{w'v,c}$, and so $h_a^* = t^{-1}(h_b^*)$. This proves that the family (h_a^*) is a natural trasformation from A to B (in fact the smallest one). ▨

II.7. remark

We can reformulate the construction of theorem II.6. as an inductive process. Given any ordinal α , we define simultaneously a family (h_a^α) of weak morphisms from A to B at stages a, where a varies through all finite dimensional ptykes a of type τ :

h_a^α is the smallest weak morphism at stage a greater than all $t^{-1}(h_b^\beta)$, where β varies through all ordinals $< \alpha$, b varies through all finite dimensional objects of $|\tau|$, and t through all elements of I(a,b).

There must be some stage α where one of the following holds :

- either some h_a^α does not exist (i.e. is not total) : then there is no natural transformation from A to B.
- or for all a finite dimensional, $h_a^\alpha = h_a^{\alpha+1}$; then $t^{-1}(h_b^\alpha) = h_a^\alpha$ for all b finite dimensional and all $t \in I(a,b)$. In that case (h_a^α) defines a natural trasformation from A to B, since a natural transformation is determined by its action on finite dimensional objects. Obviously $h_a^\alpha = h_a^*$.

The sets $A = \{(a,i,j); h_a(i) \geq j\}$ are clearly given by a positive inductive definition, and in particular, when all data are recursive, this definition will close not later than level ω_1^{CK}. So in that case, it takes at most ω_1^{CK} steps to compute the family (h_a^*).

II.8. remark

It is of some interest to remark that in the question of weak morphisms from A to B, very little is indeed required from A :

(i) the fact that A(a) is linearily ordered is never used

(ii) the fact that A preserves pull-backs is not really used ; of course one must give a sense to $\mathrm{Den}_{A,a}(z)$ when A does not preserve pull-backs. By definition, this is the intersection of all sets rg(Tr(t)), where t varies through all morphisms whose target is a, and such that $z \in \mathrm{rg}(A(t))$.

Lemma IV.7. is a typical use of this remark.

III FUNCTORIAL BOUNDEDNESS

The general aim of this section is to link the notion of definability with the notion of functoriality. More precisely, to show that a sufficiently definable function from ptykes of type σ to ptykes of type τ can be replaced by a recursive ptyx of type $\sigma \to \tau$. Now for obvious reasons, this replacement can only be achieved as a boundedness result : the function F is bounded by the ptyx Φ , in the sense of pointwise boundedness : for any argument a of type σ, F(a) is bounded by $\Phi(a)$. If $\tau = 0$, we have a natural definition for " $\Phi(a)$ bounds F(a)"; in general we shall use the strongest available notion of boundedness : F(a) is bounded by $\Phi(a)$ means that there is a morphism (an embedding) from F(a) to $\Phi(a)$.

The theory of amalgamation of section I is enough to obtain functorial boundedness in contexts where the arguments are denumerable ; the results of section II will enable us to extend these results to non denumerable arguments.

III.1. theorem (functorial boundedness theorem, first version)

If $A \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is a Σ^1_1 set such that for all $(\beta, \gamma) \in A$, whenever β codes a ptyx $b_\beta \in |\sigma|$, then γ codes $c_\gamma \in |\tau|$, then there is a recursive ptyx Φ of type $\sigma \to \tau$ such that for all $(\beta, \gamma) \in A$, whenever $b_\beta \in |\sigma|$, then c_γ can be embedded into $\Phi(b_\beta)$.

PROOF : we can assume that A is non void, and then A is the range of a monotoneous Souslin scheme, i.e. there is a monotoneous recursive map $\bar{F}(\bar{\alpha}(n)) = (\bar{\beta}(m), \bar{\gamma}(m))$, with an extension to a continuous map $F(\alpha) = (\beta, \gamma)$, such that $A = \{F(\alpha); \alpha \in \mathbb{N}^{\mathbb{N}}\}$. We shall use the notation

$$\bar{F}(\bar{\alpha}(n)) = (\beta_{\bar{\alpha}(n)}, \gamma_{\bar{\alpha}(n)}).$$

Without loss of generality, we may assume that $\beta_{\bar{\alpha}(n)}$ and $\gamma_{\bar{\alpha}(n)}$ code finite dimensional ptykes $b_{\bar{\alpha}(n)}$ and $c_{\bar{\alpha}(n)}$ of respective types σ and τ, as well as embeddings $u_{\bar\alpha(p)\bar\alpha(q)} \in I(b_{\bar\alpha(p)}, b_{\bar\alpha(q)})$, $v_{\bar\alpha(p)\bar\alpha(q)} \in I(c_{\bar\alpha(p)}, c_{\bar\alpha(q)})$ for all $p \leq q < n$. Hence when $F(\alpha) = (\beta, \gamma)$, $(b_\beta, u_{\bar\alpha(n)}) = \varinjlim(b_{\bar\alpha(n)}, u_{\bar\alpha(n)\bar\alpha(m)})$ and $(c_\gamma, v_{\bar\alpha(n)}) = \varinjlim(c_{\bar\alpha(n)}, v_{\bar\alpha(n)\bar\alpha(m)})$ for appropriate $u_{\bar\alpha(n)}$'s and $v_{\bar\alpha(n)}$'s.

Let a be a ptyx of type σ ; we define an index set I_a as follows :

- I_a consists of all pairs $(\bar\alpha(n), t)$ where $\bar\alpha(n)$ is a sequence of integers of length n, and $t \in I(b_{\bar\alpha(n)}, a)$
- $(\bar\alpha(n), t) \leq_{I_a} (\bar\alpha(m), t')$ iff $n \leq m$, $\bar\alpha(m)$ extends $\bar\alpha(n)$ and $t = t' u_{\bar\alpha(n)\bar\alpha(m)}$.

I_a is an optree, and can be used to index an inductive system (C_i, V_{ij}) of ptykes of type τ : let $C_{(\bar\alpha(n),t)} = c_{\bar\alpha(n)}$, $V_{(\bar\alpha(n),t)(\bar\alpha(m),t')} = v_{\bar\alpha(n)\bar\alpha(m)}$.

In order to apply theorem I.3. we must linearize I_a into some wellorder R_a. For this it suffices (if we recall the way I is extended to R in the proof of I.3.) to indicate how we compare $i' = (\bar\alpha'(n+1), t')$ and $i'' = (\bar\alpha''(n+1), t'')$ when both indices extend $(\bar\alpha(n), t)$:

- if $\alpha'(n) < \alpha''(n)$, then $i' <_{R_a} i''$
- if $\alpha'(n) = \alpha''(n)$, then we must give a way to compare the morphisms t', t'' which both belong to $I(b_{\bar\alpha'(n+1)}, a)$. If σ is a product $\sigma_1 \times \ldots \times \sigma_k$, then $t' = t'_1 \otimes \ldots \otimes t'_k$, $t'' = t''_1 \otimes \ldots \otimes t''_k$, and the comparizon of t', t'' can be reduced to the lexicogrphical comparizon of their respective components.

If σ is $\rho \to 0$, then consider $\theta' = Tr(t')$, $\theta'' = Tr(t'')$. If $Tr(b_{\bar\alpha'(n+1)})$ is tne finite set $((x_1, d_1), \ldots, (x_p, d_p))$ then $\theta'(x_i, d_i) = (x'_i, d_i)$, $\theta''(x_i, d_i) = (x''_i, d_i)$ so t' is determined by the sequence $(x'_1, \ldots, x'_p)$ and t'' by the sequence $(x''_1, \ldots, x''_p)$; so we can compare t' and t'' by means of a lexicographical ordering of the sequences $(x'_1, \ldots, x'_p)$ and $(x''_1, \ldots, x''_p)$.

We define $\phi(a) = \operatorname{amalg}_{R_a}(C_i, V_{ij})$. $\phi(a)$ is a ptyx of type τ ; moreover if $\alpha \in \mathbb{N}^{\mathbb{N}}$, if $F(\alpha) = (\beta, \gamma)$, if $b_\beta \in |\sigma|$, consider the linear subset I of I_{b_β} consisting of all indices $(\bar\alpha(n), u_{\bar\alpha(n)})$. Then $c_\gamma = \varinjlim_{I}(C_i, V_{ij})$, and by the universality of direct limits, there must be an embedding from c_γ to $\phi(b_\beta)$.

Assume now that $w \in I(a,a')$, $a,a' \in |\sigma|$; then there is a function h_w from $|I_a|$ to $|I_{a'}|$, defined by $h_w((\bar\alpha(n),t)) = (\bar\alpha(n),vt)$. The way we linearized I_a into R_a is "functorial" : this means that h_w is morphism (in the category IND^τ) from $(I_a,R_a,(C_i^a,V_{ij}^a))$ to $(I_{a'},R_{a'},(C_i^{a'},V_{ij}^{a'}))$, so we can define $\Phi(w) = \underset{\vec{h}_w}{\mathrm{amalg}}(I_{a'},R_{a'},(C_i^{a'},V_{ij}^{a'}))$. By I.7.(ii) , the functor Φ will preserve direct limits and pull-backs (we use the fact that the functor $\Psi(a) = (I_a,R_a,(C_i^a,V_{ij}^a))$, $\Psi(w) = h_w$ already preserves direct limits and pull-backs).

Finally we must check that Φ is recursive ; but remark I.7.(iv) shows that, for a finite dimensional, $\Phi(a)$ is recursive in the parameter a. So Φ is a recursive ptyx of type $\sigma\to\tau$ and we are done. ☒

III.2. corollary (Kechris & Woodin, [2])

If A is a Σ_1^1 set of denumerable ptykes of type τ, then one can find a recursive $a \in |\tau|$ such that any element of A can be embedded into a.

PROOF : apply theorem III.1. with $\sigma = ()$. ☒

III.3. theorem (functorial boundedness theorem, second version)

Let F be a set-recursive function and assume that whenever $a \in |\sigma|$, then $F(a) \in |\tau|$; then there is a recursive ptyx Φ of type $\sigma \to \tau$ such that for all $a \in |\sigma|$, F(a) is embeddable in $\Phi(a)$.

PROOF : by induction on τ : the theorem is easily reduced to the case $\tau = \rho \to 0$. There is a Σ_1^1 relation $A \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that, whenever β and γ code sets x_β and y_γ, and $F(x_\beta)$ is defined, then

$$y_\gamma = F(x_\beta) \quad \leftrightarrow \quad (\beta,\gamma) \in A$$

Theorem III.1. applied to this set A, yields a recursive ptyx Φ of type $\sigma \to \tau$ that works for all countable a.

Now assume that Φ doesn't work in the uncountable case : for some a, F(a) is not embeddable in $\Phi(a)$. Now, by theorem II.6., this means that there is some $b \in |\rho|$ such that there is no weak morphism from F(a) to $\Phi(a)$ at stage b. By a Löwenheim-Skolem argument, there will be denumerable subobjects $\bar a$ (of a) and $\bar b$ (of b) such that there is already no weak morphism from $F(\bar a)$ to $(\bar a)$ at stage $\bar b$. (This uses remark II.3.) Contradiction ☒

III.4. remark

In [3], there is an application of a more general form of theorem III.3. . The general form has exactly the same proof.

IV T_A - THEOREMS

IV.1. definition

Let σ be a type of the form $\tau \to 0$, and let A be a ptyx of type σ (more generally a preptyx) ; for each $a \in |\tau|$, we define a tree $T_A(a)$ as follows : $T_A(a) = \{(z_0,\ldots,z_n); n \in \mathbb{N} \ \& \ z_0,\ldots,z_n \in A(a) \ \& \ z_n < \ldots < z_0\}$
When $t \in I(a,b)$, we define $T_A(t)$, a function from $T_A(a)$ to $T_A(b)$ by :

$$T_A(t)(z_0,\ldots,z_n) = (A(t)(z_0),\ldots,A(t)(z_n))$$

In general, it is easier to work with the trees T_A than with A itself ; in particular, in many situations, we shall get bounds (w.r.t. embeddability) on T_A, that are not easily transformable into bounds on A. The following theorem, a corollary to the results of III, shows that this is however possible.

IV.2. theorem

If $\sigma = \tau \to 0$, one constructs a recursive ptyx Φ^σ of type $\sigma \to \sigma$ such that :
if T_A can be embedded into the ptyx B of type σ, then A can be embedded into $\Phi^\sigma(B)$.

PROOF : what means "T_A can be embedded into B" ? This means that for all $a \in |\tau|$, one has a function θ_a from $T_A(a)$ to $B(a)$, which is orderpreserving, and that when $t \in I(a,b)$, $\theta_b T_A(t) = B(t)\theta_a$. (A typical example of such an embedding is when we linearize T_A into a ptyx by means of the Kleene-Brouwer ordering).

By the Lowenheim-Skolem argument used in III.3., it is sufficient to construct a Φ^σ that will work for countable A and B. Let R be the relation defined between elements of $\mathbb{N}^{\mathbb{N}}$:

$(\beta,\gamma) \in R$ iff β codes a preptyx B, γ codes a preptyx A, and T_A can be embedded into B.(A,B of type σ)

Then R is such that theorem III.1. can be applied, and this yields our Φ^σ . (More precisely, in order to apply the functorial boundedness theorem we must check that R is Σ^1_1 (obvious), and that whenever T_A can be embedded into a ptyx B, then A is a ptyx : but the embeddability of $T_A(a)$ into $B(a)$ implies that $T_A(a)$ is well-founded, which in turn implies that $A(a)$ is a wellorder.) ☒

IV.3. remark

The T_A-technology is extremely efficient. For instance it is easy to prove the functorial boundedness theorem from theorem IV.2. : we sketch the argument, assuming that τ is $\rho \to 0$.

(i) Construct a $\sigma X\rho$-theory T (analogue of a β-theory ([1],chapter X) obtained by replacing ON by the category $\sigma X \rho$), containing among its symbols letters $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$, and $\underline{\theta}$; the axioms for $T[a \otimes d]$ express that :

- $F(\underline{\alpha}) = (\underline{\beta}, \underline{\gamma})$

- $\underline{\theta}$ is an embedding from b_{β} into a

(ii) then by a completeness argument, one obtains a functorial recursive $\sigma X\rho$-proof π : $\pi(a \otimes d)$ proves that $c_{\underline{\gamma}}(d)$ is a wellorder. Now observe that (up to inessential details) $\pi(a \otimes d)$ must contain a copy of $T_{c_\gamma}(d)$ for any α such that b_{β} can be embedded into a.

(iii) let π' be any recursive ptyx linearizing π (for instance by some functorial Kleene-Brouwer ordering of the branches) ; then $\pi' \circ \Phi^{\rho \to 0}$ answers the problem.

The reader can also prove the amalgamation result of I.3., considering a τ-theory in which we can prove that "B(a) has no s.d.s.", functorially in a ...

However, the effect of this remark is limited, since we don't know any direct proof of the T_A-theorem IV.2. (except for dilators, see IV.6.).

IV.4. theorem

Let $A \subset \mathbb{N}^{\mathbb{N}}$ be a Σ^1_k set of (codes for) ptykes of type σ . Then we can form a recursive ptyx of type $k\text{-}2 \to \sigma$,say Ψ such that :

every ptyx (encoded by a point) of A can be embedded into some $\Psi(d)$, for some denumerable $d \in |k\text{-}2|$.

PROOF : in the theorem k-2 may take the values -1,0,1,... ; the types denoted by these integers are (),0,0→0,... in general n+1 = n→0. The result for $k = 1$ is just Kechris & Woodin's III.2., hence we shall assume that $k \geq 2$. We shall also assume that σ is $\tau \to 0$.

By hypothesis, for $\alpha \in A$, α codes a ptyx a_α of type σ. Also, using Π^1_k-completeness of ptykes of type k-1, there is a recursive function F with the property that

$$\alpha \notin A \leftrightarrow c_{F(\alpha)} \in |k-1|$$

where $c_{F(\alpha)}$ is the preptyx of type k-1 encoded by $F(\alpha)$.

For $\alpha \in \mathbb{N}^{\mathbb{N}}$, $b \in |\tau|$, $d \in |k-2|$, form a tree $T^\alpha(d,b)$:

$$T^\alpha(d,b) = \{(x_0,y_0,\ldots,x_n,y_n); x_n < \ldots < x_0 \text{ in } a_\alpha(b) \;\&\; y_n < \ldots < y_0 \text{ in } c_{F(\alpha)}(d)\}$$

Now it is immediate that $T^\alpha(d,b)$ is always well-founded. We can linearize the functorial trees T^α into ptykes D^α of type $k-2 \to \sigma$. Now by III.2., there is a recursive ptyx D of the same type such that all D^α's can be embedded into D.

Now, consider $\alpha \in A$; then $c_{F(\alpha)} \notin |k-1|$, so there is a denumerable $d \in |k-2|$ and a s.d.s. (y_n) in $c_{F(\alpha)}(d)$. Then observe that we can embed the functorial tree T_{a_α} into $T^\alpha(d,.)$ as follows : define $\theta(b)$ by

$$\theta(b)(x_0,\ldots,x_n) = (x_0,y_0,\ldots,x_n,y_n)$$

By composition of embeddings, we can embed T_{a_α} into $D^\alpha(d)$, and into $D(d)$. But then a_α can be embedded into $\phi^\sigma(D(d))$. Hence the theorem is proved with $\Psi = \phi^\sigma \circ D$ ⊠

IV.5. remarks

(i) For instance, if A is a Σ^1_2 set of dilators, there is a recursive dilator of two variables D such that any element of A can be embedded into some D(.,y). This is clearly a sharpening of a result stated in [2] saying that in such a situation, one can find a nondenumerable D in which all the elements of the set are embeddable.

(ii) Again in the case of dilators, if A is Σ^1_2 , then one can find a recursive dilator D such that any element of A can be embedded into some predecessor of D : starting with D as in (i), it can be assumed that D is a bilator, then formulation (ii) works with UN(D).

(iii) if A is Σ^1_{k-1}, then d can be found uniformly recursively in α. This is left to the reader.

IV.6. theorem

In the case of dilators, we can give an explicit value for Φ^σ:

$\Phi^{0+0}(B) = (\underline{2}+\mathrm{Id})^B \circ ((\underline{\omega+1}).\mathrm{Id} + \underline{\omega})$

PROOF : i) assume that T_A is embeddable into B, and let A* be the quasi-dendroid associated with A ([1], chapter 8). A*(a) is the set of extremal nodes of a functorial tree, equipped with Kleene-Brouwer ordering. Let us recall that the sequences in A*(a) are of the form $(x_0,\dots,x_{2n})$, and that the "mutilation" functions A*(t) are defined by

$A^*(t)(x_0,\dots,x_{2n}) = (x_0,t(x_1),x_2,t(x_3),\dots,x_{2n})$.

For obscure technical reasons, we introduce A** : A**(a) is the set of all sequences $(x_0,\dots,x_{2i})$ which have some extension in A*(a). The ordering of A**(a) is a bit uneven : $(x_0,\dots,x_{2i}) \leq (y_0,\dots,y_{2j})$ iff $i \geq j$, $x_0 = y_0,\dots,x_{2j-1} = y_{2j-1}, x_{2j} \leq y_{2j}$.

IV.7. lemma

If T_A is embeddable into B, then A** is embeddable into $C = B \circ ((\underline{\omega+1})\mathrm{Id} + \underline{\omega})$.

PROOF : let $U = T_A \circ ((\underline{\omega+1})\mathrm{Id} + \underline{\omega})$; for $a \in On$, let h_a be the smallest weak morphism from U to C at stage a. (Here observe that the results of II on weak morphisms will still hold, as long as we keep the linearity of the targets, here C(a)). If $s = (x_0,\dots,x_{2i}) \in A^{**}(a)$, extend s into some $s' = (x_0,\dots,x_{2n}) \in A^*(a)$, and define $t = (y_0,\dots,y_{2n})$ by :

$y_0 = x_0, y_2 = x_2,\dots,y_{2n} = x_{2n}; y_1 = (\omega+1)x_1 + \omega,\dots, y_{2i-1} = (\omega+1)x_{2i-1} + \omega$;

the values $y_{2i+1}, y_{2i+3},\dots,y_{2n-1}$ are chosen in such a way that :

+ the sequences $x_1,x_3,\dots,x_{2n-1}$ and $y_1,y_3,\dots,y_{2n-1}$ have the same order type

+ $y_{2i+1},\dots,y_{2n-1}$ are of one of the following forms :

- $(\omega+1)x_{2j+1} + p$ $\qquad (j < i,\ p \in \mathbb{N})$
- $(\omega+1)a + p$ $\qquad (p \in \mathbb{N})$

We define a function k_a from A**(a) to C(a), by $k_a(s) = h_a((t))$. (Here we have identified t with a point in $A((\omega+1)a + \omega)$, so (t) denotes a point in U(a))

First observe that $\mathrm{Den}_{U,a}(t) = \mathrm{Den}_{A^{**},a}(s)$. (A^{**} may not commute to pull-backs but $\mathrm{Den}_{A^{**},a}(s)$ is well-defined by II.8., and is the set $\{x_1,x_3,\ldots,x_{2i-1}\}$.) Assume that $s < s_1$ in $A^{**}(a)$; choose s_1', t_1 corresponding to s_1, and so we define $k_a(s_1) = h_a((t_1))$. Now observe that $t < t_1$ (proof : if s extends s_1, and $s_1 = (x_0,\ldots,x_{2j})$, then the coefficients of order $2j+1$ of t and t_1 are respectively $(\omega+1)x_{2j+1}+\omega$ and $(\omega+1)u + p$, with $u > x_{2j+1}$, so $t < t_1$; if they differ at index $2j$, then the same will be true for t and t_1, and we shall have $t < t_1$. $\blacksquare$) From this it follows that $(t_1,t) \in U(a)$; furthermore $\mathrm{Den}_{U,a}((t_1,t)) = \mathrm{Den}_{U,a}(t)$ (because $\mathrm{Den}_{A^{**},a}(s_1)$ $\mathrm{Den}_{A^{**},a}(s)$). Now, since h_a is the smallest weak morphism at stage a, it follows that $h_a((t_1,t) * u) = h_a((t) * u)$. (proof : the non-trivial property is that $h_a((t_1,t) * u) \geq h_a((t) * u)$: change h_a into h_a' by replacing the values on $(t) * u$: $h_a'((t) * u) = h_a((t_1,t) * u)$. This is possible because $\mathrm{Den}_{U,a}((t) * u) = \mathrm{Den}_{U,a}((t_1,t) * u)$. Then use the minimality of h_a. $\blacksquare$) From this $h_a((t_1)) > h_a((t))$.

Then k_a is a weak morphism from A^{**} to C at stage a. By the general results of section II (and remark II.8.) A^{**} can be embedded into C. $\blacksquare$

ii) We have got an embbedding T from A^{**} into C. Then we shall embed A into $(\underline{2}+\mathrm{Id})^C$, by defining : $T'(a)((x_0,\ldots,x_{2n})) =$

$$(2+a)^{T(a)((x_0))}.(2+x_1)+\ldots\ldots+(2+a)^{T(a)((x_0,\ldots,x_{2n-2}))}.(2+x_{n-1})+(2+a)^{T(a)(x_0,\ldots,x_{2n})}.$$

First observe that the expression makes sense :

$T(a)((x_0,\ldots,x_{2n})) < T(a)((x_0,\ldots,x_{2n-2})) < \ldots < T(a)((x_0))$

because $(x_0,\ldots,x_{2n}) < (x_0,\ldots,x_{2n-2}) < \ldots < (x_0)$ in $A^{**}(a)$.

Next, observe that $T'(a)$ is strictly increasing : if $(x_0,\ldots,x_{2n}) < (y_0,\ldots,y_{2m})$ and the coefficients first differ at index

+ $2i+1$: then the comparizon of the Cantor Normal forms is immediate
+ $2i$: here we need $T(a)((x_0,\ldots,x_{2i})) < T(a)((y_0,\ldots,y_{2i}))$;

but $(x_0,\ldots,x_{2i}) < (y_0,\ldots,y_{2i})$ for the order $A^{**}(a)$.

Finally it is clear that $T'(.)$ enjoys the ad hoc commutative diagrams, so $T' \in I(A,(2+\mathrm{Id})^C)$. This concludes the proof. $\blacksquare$

IV.8. remark

The situation of theorem IV.6. shows a great difference between dilators and ptykes : for ptykes we construct effective solutions, but they are obtained by rather abstract considerations. For dilators we have the dendroidal representation which gives us a precious explicit geometrical insight to dilators. We have a reasonable understanding of dilators, but we must admit that general ptykes are just convenient abstractions. In particular the development of a geometrical structure of ptykes would be a great progress; a possibility is that the concept of ptyx is too general, and that additional preservation properties must be found.

Among the open questions connected with ptykes, let us mention the following : find a ptyx F^σ of type $\sigma \to \sigma$ together with a morphism $T \in I(ID^\sigma, F^\sigma)$ such that :

given $a,b,c \in |\sigma|$, $t \in I(a,b)$, $u \in I(c,F^\sigma(b))$, with c finite dimensional, then one can find $v \in I(c,F^\sigma(a))$, such that

(i) $F(t)v \leq u$

(ii) if d, $w \in I(d,c)$ are defined by $F^\sigma(t)$ & $\dot{u} = \dot{u}w$, then $vw = \dot{u}w$

The formulation may seem rather obscure, but the property is very natural : for instance, if A is a functor from σ to τ preserving direct limits, then $A \circ F^\sigma$ is a ptyx of type $\sigma \to \tau$ (Only part (ii) of the property is used to obtain preservation of pull-backs). In the case $\sigma = 0$, one knows a solution, namely $F^0 = \underline{\omega}(\underline{1}+Id)$.

If $a' = F^\sigma(a)$, if $t' = T(a)$ (so $t' \in I(a,a')$, then given any finite dimensional $b,b' \in |\sigma|$, $v \in I(b,a)$, $v' \in I(b,b')$, then it is possible to find $u' \in I(b',a')$ such that $u'v' = t'v$. This weaker property is an easy consequence of (ii). Using amalgamation it is easy to construct a' enjoying this weaker version. But it is not known how to make a' a functor of a enjoying requirements (i),(ii).

APPENDIX : ON A QUESTION OF KECHRIS

In [4], Kechris asked a question for which we give here a solution. The solution uses a double-tree method, analogous to the one used for theorem IV.4., and which is a variant of the mollification method used in the theory of continuous functionals.

V.1. theorem

There is a recursive dilator D such that the set

$\{ D'(\omega_1) ; D' \text{ is embeddable in } D\}$

is not denumerable.

PROOF : let LORD be the set of all linear orderings of $\mathbb{N}$. For $\alpha \in$ LORD, $x \in$ On we let $T^\alpha(x)$ be the tree of sequences $(x_0, k_0, \ldots, x_{n-1}, k_{n-1})$, where $(k_0, \ldots, k_{n-1})$ is a strictly descending sequence in α and $\tau(i) = x_i$ is an order-preserving map from $\alpha \restriction \{0, \ldots, n-1\}$ to x.

For obvious reasons, $T^\alpha(x)$ is always well-founded ; by Kleene-Brouwer linearization, T^α can be transformed into a dilator D^α, and by III.2. there is a recursive D such that all D^α's are embeddable into D.

V.2. lemma

If α is a wellorder, then $D^\alpha(\omega_1) \leq \omega_1^{\|\alpha\|+1}$

PROOF : consider the well-founded tree

$S = \{(x_0, k_0, \ldots, x_{n-1}, k_{n-1}) ; x_0, \ldots, x_{n-1} < \omega_1$ and $(k_0, \ldots, k_{n-1})$ is a strictly descending sequence in $\alpha\}$. Let $\sigma = (x_0, k_0, \ldots, x_{n-1}, k_{n-1}) \in S$, and let $S_\sigma = \{\tau ; \sigma * \tau \in S\}$. By induction on $z = \| k_{n-1} \|_\alpha$ we show that $\|S_\sigma\| \leq \omega_1^{z+1}$. The induction is easy and left to the reader. We get $\|S\| \leq \omega_1^{\|\alpha\|+1}$, and from this the lemma follows. ∎

V.3. lemma

Let α be a wellorder and assume that $\|\alpha\|$ is a limit ordinal $\omega.z$. Then $D^\alpha(\omega_1)+1 \geq \omega_1^z$.

PROOF : say that a sequence $(x_0, \ldots, x_{n-1})$ is α-consistent when it has an extension to an embedding from α to ω_1. We let

$T' = \{(x_0,k_0,\ldots,x_{n-1},k_{n-1}) \in T^\alpha(\omega_1);(x_0,\ldots,x_{n-1}$ is consistent$\}$.

Now, let $\sigma = (x_0,k_0,\ldots,x_{n-1},k_{n-1}) \in T'$ and assume that $\|k_{n-1}\|_\alpha = \omega.z_\sigma$ for some $z_\sigma \leq z$. By induction on z , we will prove that $\|T'_\sigma\| + 1 \leq \omega_1^z$. (The "+1" only matters when $z = 0$!). If $z_\sigma = 0$, then T'_σ is empty, so $\|T'_\sigma\| = 0$, $\omega_1^{z}\sigma = 1$. Now assume that $z_\sigma > 0$, and let $u < z_\sigma$. Let t be the smallest number $\geq n$ such that $i <_\alpha t$ for all $i < t$. Since $\|\alpha\|$ is a limit ordinal, there will be such a number t. Let $x_n,\ldots,x_{t-1}$ be any α-consistent continuation of $(x_0,\ldots,x_{n-1})$ and let $k_n,\ldots,k_{t-1}$ continue the descending sequence in such a way that $\|k_{t-1}\| > \omega.u$. Now, there are ω_1 ways to continue the sequence with a consistent x_t and k_t such that $\|k_t\| = \omega.u$. For each such continuation τ , the subtree T'_τ has ordertype $\geq \omega_1^u$ (-1), hence the subtree $T'_{(x_0,k_0,\ldots,x_{t-1},k_{t-1})}$ will have ordertype $\geq \omega_1^{u+1}$.

Since $u < z_\sigma$ was arbitrary, the full tree T'_σ must have an ordertype $\geq \omega_1^{z}\sigma$, and this ends the proof of the lemma. ∎

The two lemmas show that for any countable x, there is a dilator D^α such that $D^\alpha(\omega_1) \in [\omega_1^x, \omega_1^{\omega . x+1}[$ and the theorem is proved.

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